

REMARKS ON THE DYNAMIC OF THE RUELLE OPERATOR AND INVARIANT DIFFERENTIALS.

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ABSTRACT. Let R be a rational map. We are interesting in the dynamic of the Ruelle operator on suitable spaces of differentials. In particular the necessary and sufficient conditions (in terms of convergence of sequences of measures) of existence of invariant conformal structures on $J(R)$ are obtained.

Introduction and main statements

Let R be a rational map. Let $c \in J(R)$ be a critical point with infinite forward orbit which does not contain any other critical point. Then formally we have the following possibilities:

- (1) $\lim_{n \rightarrow \infty} |(R^n)'(R(c))| = 0$,
- (2) there exists a subsequence $\{n_i\}$ such that $\lim_{i \rightarrow \infty} |(R^{n_i})'(R(c))| = \infty$,
- (3) there exists a subsequence $\{n_i\}$ such that $\lim_{i \rightarrow \infty} |(R^{n_i})'(R(c))| = M < \infty$ and $M \neq 0$.

We believe that the first case contains a contradiction. Because in this situation the forward orbit of c must converge to an attractive (superattractive) cycle and hence $c \notin J(R)$. This conjecture is true for example for real quadratic polynomials whose critical point has strictly negative Lyapunov exponent (see [MS]).

As for the last two cases, the Fatou conjecture claims that R is an unstable map. With additional conditions placed on the behavior of Poincaré series (see definitions below) we know that in this situation R is an unstable map see ([Av, Mak2]) for maps with non-empty Fatou set ([Av]) and for more general situation ([Mak2]) and ([Lev, Mak1]) for polynomials of degree two. By using arguments of ([Av, Lev, Mak1, Mak2]) we reproduce this result for any rational map (see theorem 16). How we know this conditions appear independently in works of P. Makienko [Mak1] and G. Levin [Lev]. Unfortunately we can not avoid these additional conditions even in the best case (the Poincaré series is absolutely convergent) see ([Av, Lev, Mak2]) in this case these conditions are connected with the following conjecture (see below "Generalized Sullivan Conjecture"):

1991 *Mathematics Subject Classification.* Primary:37F45, Secondary:37F30..

This work has been partially supported by the Russian Fund of Basic Researches, Grant 99-01-01006. This work was partially supported by proyecto **CONACyT** *Sistemas Dinámicos* 27958E from Mexico

Except for the Lattés maps, there is no measurable invariant integrable (over $\overline{\mathbb{C}}$) quadratic differential for rational maps.

The infinitesimal content of Thurston's Uniqueness Theorem (see ([DH] and also [MM1]) is the assertion $R^*(\phi) \neq \phi$ for non-zero integrable meromorphic quadratic differential ϕ , where R^* is an "pushforward" or "Ruelle" operator (see definition below) associated with the rational map R . C. McMullen ([MM2]) proves that R is the Lattés map if and only if there exists an invariant integrable meromorphic differential. A. Epstein ([E]) extends the results above to: there is no invariant meromorphic differential on $\overline{\mathbb{C}}$ for any rational map except the Lattés map. P. Makienko ([Mak2]) shows that with additional assumptions on postcritical set there is no invariant integrable differential among the augmented meromorphic differentials except the Lattés map. The differential ϕ is *augmented meromorphic* if $\phi = \sum_{i \geq 0} \gamma_i$, where γ_i are meromorphic integrable differentials with only four simple poles: three of them are $0, 1, \infty$ and series $\sum \|\gamma_i\|$ is absolutely convergent.

Let us call a measurable differential ϕ as a *regular* iff $\overline{\partial}\phi$ is a finite complex valued measure on \mathbb{C} , here $\overline{\partial}$ in distributions. Note that any augmented differential is a regular.

In this paper we extend the result from [Mak2] and show that with the same assumption (like in [Mak2]) on the postcritical set there is no invariant regular differential except the Lattés map. (Theorem A).

The next two results (see propositions 14-16) give the necessary and sufficient conditions (in the terms of dynamic of the Ruelle operator and its modulus) to absence of the invariant conformal structures on the Julia set and to equality to 0 the Lebesgue measure of the Julia set, respectively.

In our final results (theorems B and C) we reformulate these mentioned additional conditions in other terms such as the common behavior of the Ruelle-Poincaré series (theorem C) and convergence of the measures's sequence (theorem B).

The main technical idea is:

Assume that $J(R)$ supports a non - trivial invariant conformal structure μ . Let f_μ be its corresponding quasiconformal map. We find the conditions which allow us to construct a quasiconformal map h supported already on the Fatou set, so that h and f_μ generate the same infinitesimal deformation of R (see also [Mak]).

Definitions and Main results.

Let R be a rational map.

Denote by $P(R) = \overline{\{\cup_c \cup_n R^n(c), c \text{ is a critical point}\}}$ the postcritical set of R .

Start again with a rational map R and consider two actions $R_{n,m}^*$ and $R_{*n,m}$ on a function ϕ at point z by the formulas

$$R_{n,m}^*(\phi) = \sum \phi(J_i)(J_i')^n (\overline{J_i'})^m = \sum_{y \in R^{-1}(z)} \frac{\phi(y)}{(R'(y))^n (\overline{R'(y)})^m}, \text{ and}$$

$$R_{*n,m}(\phi) = \phi(R) \cdot (R')^n \cdot \overline{(R')^m},$$

where n and m are integers and $J_i, i = 1, \dots, d$ are branches of the inverse map R^{-1} . Then we have

$$R_{n,m}^* \circ R_{*n,m}(\phi) = \deg(R) \cdot \phi.$$

In other words, the actions above consist of the natural actions of R on the spaces of forms of type $\phi(z)Dz^mD\bar{z}^n$.

Definition.

- (1) The operator $R^* = R_{2,0}^*$ is called the transfer (pushforward) operator or Ruelle operator of the rational map R .
- (2) The operator $|R^*| = R_{1,1}^*$ is called the modulus of the Ruelle operator
- (3) The operator $B_R = R_{*-1,1}$ is called the Beltrami operator of the rational map R .

The operators R^* and $|R^*|$ and their right inverses $R_*(\phi) = \frac{R_{*2,0}(\phi)}{\deg(R)}$ and $|R_*|(\phi) = \frac{|R_{*2,0}|(\phi)}{\deg(R)} = \frac{\phi(R)|R'|^2}{\deg(R)}$ map the space $L_1(\bar{\mathbb{C}})$ into itself with the unit norm. Moreover the operator B_R maps the space $L_\infty(\bar{\mathbb{C}})$ into itself with unit norm.

Generalized Sullivan Conjecture. *Except for the Lattés map, there are no integrable supported on the Julia set measurable functions which are invariant with respect to any of the operators enumerated above.*

L_p -topology. Assume $1 < p < \infty$, then we can define the action of the Ruelle operator $(R_*)_p : L_p(\bar{\mathbb{C}}) \mapsto L_p(\bar{\mathbb{C}})$ by the following way:

$$(R_*)_p(\phi) = \frac{1}{d^{\frac{p-1}{p}}} \sum \phi(J_i)(J'_i)^{\frac{2}{p}}.$$

By analogy with the L_1 case we define the *modulus* of $(R_*)_p$ by the following formula:

$$|(R_*)_p|(\phi) = \frac{1}{d^{\frac{p-1}{p}}} \sum \phi(J_i)|(J'_i)|^{\frac{2}{p}}.$$

For definition and properties of the Lattés maps we are referee to the preprint of J. Milnor (see [M]). Follow by J. Milnor we say that a Latté map R is *flexible* iff $J(R)$ posses a non trivial fixed point of the Beltrami operator B_R .

Proposition A.

- (1) A rational map R is a flexible Latté map if and only if there exists a $p > 1$, such that $(R_*)_p$ has a fixed point in $L_p(\bar{\mathbb{C}})$.
- (2) A rational map R is a Latté map if and only if there exists a $p > 1$, such that the modulus $|(R_*)_p|$ has a fixed point in $L_p(\bar{\mathbb{C}})$.

Proof. Proof follows from results of A. Zdunik ([Zd]) and mean ergodicity lemma (see below).

Definition. Let $R \in \mathbb{C}P^{2d+1}$ be a rational map. The component of **J**-stability of R is the following space.

$$qc_J(R) = \left\{ F \in \mathbb{C}P^{2d+1} : \text{there are neighborhoods } U_R \text{ and } U_F \text{ of } \mathbf{J}(R) \text{ and } \mathbf{J}(F), \right. \\ \left. \text{respectively and a quasiconformal homeomorphism } h_F : U_R \rightarrow U_F \text{ such that} \right. \\ \left. F = h_F \circ R \circ h_F^{-1} \right\} / PSL_2(\mathbb{C}).$$

Definition. Let R be a rational map. Then the space of invariant conformal structures or invariant line fields or Teichmüller space $T(J(R))$ of $J(R)$ is the following space

$$T(J(R)) = \{ \text{Fix}(B_R)|_{L_\infty(J(R))} \}.$$

where $\text{Fix}(B_R) \subset L_\infty(J(R))$ is the space of invariant functions for the Beltrami operator B_R . A result of D. Sullivan (see [S]) gives a bound of $2\deg(R) - 2$ for the dimension of $T(J(R))$.

Definition. Ruelle-Poincaré series.

- (1) Backward Ruelle-Poincaré series.

$$RS(x, R, a) = \sum_{n=0}^{\infty} (R^*)^n(\tau_a)(x),$$

where $\tau_a(z) = \frac{1}{z-a}$ and $a \in \overline{\mathbb{C}}$ is a parameter. The series

$$S(x, R) = \sum_{n=0}^{\infty} |R^*|^n(\mathbf{1}_{\mathbb{C}})(x)$$

is called the Backward Poincaré series.

- (2) Forward Ruelle-Poincaré series.

$$RP(x, R) = \sum_{n=0}^{\infty} \frac{1}{(R^n)'(R(x))}.$$

The series

$$P(x, R) = \sum_{n=0}^{\infty} \frac{1}{|(R^n)'(R(x))|}$$

is called the forward Poincaré series. The series

$$A(x, R, a) = \sum_{n=0}^{\infty} \frac{1}{(R^n)'(a)(x - R^n(a))}$$

is called the modified Ruelle-Poincaré series.

Note that the Ruelle-Poincaré series generalize the Poincaré series introduced by C. McMullen for rational maps (see [MM]).

Definition. We call a measurable integrable function ϕ a regular iff $\overline{\partial}\phi$ is a complex-valued finite measure on $\overline{\mathbb{C}}$, here derivative in the sense of distributions.

Definition. Let X be the space of rational maps R that satisfy one of the following conditions

- (1) the diameters of all components of $\mathbb{C} \setminus P(R)$ are uniformly bounded away from zero or
- (2) $m(P(R)) = 0$, where m is the Lebesgue measure or
- (3) $\{P(R) \cap J(R)\} \subset \cup \partial D$, where union runs over components $D \subset F(R)$.

Note that the last condition holds for maps with completely invariant domain.

Theorem A. Let $R \in X$ be a rational map. Then either R is a Lattés map or there is no regular fixed points for $R^* : L_1(\overline{\mathbb{C}}) \mapsto L_1(\overline{\mathbb{C}})$.

Now, we give necessary and sufficient conditions for the existence of measurable invariant conformal structures on the Julia set in the terms of especial sequences of measures.

Let U be a neighborhood of $J(R)$. We say that U is an *essential neighborhood* iff

- (1) U does not contain disks centered at all attractive and superattractive points and
- (2) $R^{-1}(U) \subset U$.

Definition. Let us define the space $H(U) \subset C(\overline{U})$, where $C(\overline{U})$ is the space of continuous functions and

- (1) U is an essential neighborhood of $J(R)$ and
- (2) $H(U)$ consists of $h \in C(\overline{U})$ such that $\frac{\partial h}{\partial \bar{z}}$ (in the sense of distributions) belongs to $L_\infty(U)$
- (3) $H(U)$ inherits the topology of $C(\overline{U})$.

Measures ν_l^i .

Here we assume that $0, 1$ and ∞ are fixed points for R .

- (1) Let c_i and d_i be the critical points and critical values, respectively. Let $\gamma_a(z) = \frac{a(a-1)}{z(z-1)(z-a)}$. Then define $\mu_n^i = \frac{\partial}{\partial \bar{z}}((R^*)^n(\gamma_{d_i}(z)))$ (in sense of distributions). We will show below that $(R^*)^n(\gamma_{d_i}(z)) = \sum_{j=1}^{2\deg(R)-2} \sum_{j=0}^n \alpha_j^i \gamma_{R^j(d_i)}(z)$ and hence $\mu_n^i = \sum_{j=1}^{2\deg(R)-2} \sum_{j=0}^n \alpha_j^i \{(R^j(d_i) - 1)\delta_0 - R^j(d_i)\delta_1 + \delta_{R^j(d_i)}\}$, where δ_a denotes the delta measure with mass at the point a .
- (2) Define by ν_l^i the average $\frac{1}{l} \sum_{k=0}^{l-1} \mu_k^i$.

In general the coefficients α_j^i in definition above can be expressed as a combinations of elements of the Cauchy product $RP(c_i, R, d_i) \otimes RP(c_i, R)$ of Ruelle-Poincaré series.

Theorem B. Let $R \in X$ be a rational map with the simple critical points and no simple critical relations. Let $c_1 \in J(R)$ be a critical point and $d_1 = R(c_1)$ be its critical value. If there exist an essential neighborhood U and a sequence of integers $\{l_k\}$ such that the measures $\{\nu_{l_k}^1\}$ converge in the $*$ -weak topology on $H(U)$, then the map R is not structurally stable (or is unstable map).

Corollary B.

- (1) Under assumptions of the theorem B the space $T(J(R)) = \emptyset$ if and only if there exist an essential neighborhood U and a sequences of integers $\{l_k\}$ such that the measures $\{\nu_{l_k}^i\}$ converge in the $*$ -weak topology on $H(U)$ for any $i = 1, \dots, 2\deg(R) - 2$.
- (2) If a rational map $R \notin X$, then $T(J(R)) = \emptyset$ if and only if there exist an essential neighborhood U and a sequences of integers $\{l_k\}$ such that the measures $\{\nu_{l_k}^i\}$ converge to zero in the $*$ -weak topology on $H(U)$ for any $i = 1, \dots, 2\deg(R) - 2$.

Let P be a structurally stable polynomial. Consider the following decomposition

$$\frac{1}{(P^n)'(z)} = \sum \frac{b_i^n}{z - c_i},$$

where c_i are critical points of the polynomial P^n . Let

$$B_n = \sum_i |b_i^n|.$$

Now

Definition. A structurally stable polynomial P is strongly convergent iff the sums

$$B_n < C < \infty.$$

independently on n .

Remark. We note that, in some sense, any polynomial is close to be strongly convergent. Indeed the boundedness of B_n is equivalent to the boundedness of the total variation of the following measures

$$\bar{\partial} \left(\frac{1}{(P^n)'} \right) = \sum b_i^n \delta_{c_i}.$$

Let h be a polynomial then easy calculations show

$$\sum b_i^n h(c_i) = \int_{\gamma} \frac{h(z)}{(P^n)'(z)} dz \rightarrow 0 \text{ for } n \rightarrow \infty,$$

here γ is a closed Jordan curve enclosing the closure of critical points of P^n for all n .

Theorem C. Let $P \in X$ be a strongly convergent polynomial. Then P is a hyperbolic polynomial.

Finally we shortly discuss the following interesting question

How to interpretate the coefficients of a rational map from point of view of the dynamic or what is the dynamical meaning of the coefficients?

The next results clarify the dynamical meaning of the coefficients b_i^n coming from the decomposition of $\frac{1}{(P^n)'(z)}$ and give some formal relations between Ruelle-Poincaré series.

Definition. We denote the Cauchy product of series A and B by $A \otimes B$. Let us recall that if $A = \sum_{i=1} a_i$ and $B = \sum_{i=1} b_i$, then $C = A \otimes B = \sum_{i=1} c_i$, where $c_i = \sum_{j=1}^i a_j b_{i-j}$.

Proposition C1. If P is a structurally stable polynomial, then

$$B_n = \sum_{c \in Cr(P)} \frac{1}{|P''(c)|} \sum_{j=1}^{n-1} \frac{1}{|(P^{n-j-1})'(P(c))|} \sum_{P^j(y)=c} \frac{1}{|(P^j)'(y)|^2},$$

and hence we have the following formal equality

$$\sum_{n=2} B_n = \sum_{c \in Cr(P)} \frac{1}{|P''(c)|} S(c, P) \otimes P(c, P),$$

where \otimes means the Cauchy product of the series.

Proposition C2. Let R be a rational map with simple critical points. Let ∞ be a fixed point for R . Then there exist the following formal relations.

$$RP(a, R) - 1 = \sum_i \lambda^i - \sum_i \frac{1}{R''(c_i)} RS(c_i, R, a) \otimes RP(c_i, R), \text{ where } \lambda \text{ is the multiplier of } \infty$$

$$RS(x, R, a) = A(x, R, a) + \sum_k \frac{1}{R''(c_k)} A(c_k, R, a) \otimes RS(x, R, R(c_k)),$$

where c_k are the critical points of R .

Corollary C. Let P be a structurally stable polynomial. Assume that for any critical point c there exists a constant M_c so that

$$\frac{1}{|(P^n)'(P(c))|} \leq \frac{M_c}{n} \text{ and } |R^*|^n(\mathbf{1}_C)(c) \leq \frac{M_c}{n},$$

then P is a strongly convergent polynomial.

The last section contains a little discussion and open question with respect to spectrum of the Ruelle operator and related topics.

This paper is based on the preprint ([Mak1]).

Acknowledgement. I would like to thank to IMS at SUNY Stony Brook FIM ETH at Zurich and IM UNAM at Cuernavaca for its hospitality during the preparation of this paper.

Quadratic differentials for rational maps

Let S_R be the Riemann surface associated with the action of R on its Fatou set, then (see [S]) S_R is a finite union $\cup_i S_i$ of punctured torii, punctured spheres and foliated surfaces.

Let $A(S_R)$ be the space of quadratic holomorphic integrable differentials on S_R . If $S_R = \cup_i^N S_i$, then $A(S_R) = A(S_1) \times \dots \times A(S_N)$, where $A(S_i)$ is the space of quadratic holomorphic integrable differentials on S_i .

Quadratic differentials for foliated surfaces. By results of ([S]) a foliated surface S is either unit disk or a round annulus with marked points, equipped with a group \mathbf{G}_D of rotations. This group \mathbf{G}_D is an everywhere dense subgroup in the group of all rotations of S in the topology of uniform convergence on S . Hence for any $z \in S$ the closure of the orbit $\mathbf{G}_D(z)$ yields a circle which is called *a leaf of the invariant foliation*. If the leaf l contains a marked point x , then we call l *a critical leaf* and denote it by l_x . With the exception of a single case the boundary ∂S consists of critical leaves. This exception is the surface corresponding to the full orbit of a simply connected superattractive periodic component containing only one critical point. In the latter case the surface S does not contain critical leaves. In this case, the modulus of S is not defined (see [S] for details).

Any quadratic absolutely integrable holomorphic differential ϕ must be invariant under the action of the group \mathbf{G}_D for foliated surfaces. Hence $\phi = 0$ is the only absolutely integrable holomorphic function for S with undefined modulus, and therefore we have in this case $A(S) = \{0\}$.

After removing the critical leaves from S we obtain the collection $\cup S_i \cup D$ of rings S_i and disk D (in the case of Siegel disks). We call this decomposition as *critical decomposition*. For this decomposition we have

$$\phi|_{S_i} = h_i(z) \cdot dz^2, \text{ and } \phi_D = h_0 \cdot dz^2$$

where h_0 and h_i are holomorphic absolutely integrable functions on D and S_i respectively. The calculations show that $h_i(z) = \frac{c_i}{z^2}$ and $h_0 = 0$, where c_i are arbitrary constants. From the discussion above we conclude that for a ring with k critical leaves (two of which represent the boundary of S) the dimension $\dim(A(S)) = k - 1$.

Now let S be a ring with critical decomposition $\cup_{i=1}^k S_i$ and $\phi \in A(S)$ a differential, then $\|\phi\| = 4\pi \sum_i |c_i| \text{mod}(S_i)$, where $\phi = \sum_i \phi|_{S_i} = \sum_i \frac{c_i}{z^2}|_{S_i}$ and $\text{mod}(S_i)$ is the modulus (or the extremal length of the family of curves connecting the boundary component of S_i) of the ring S_i .

We always assume here that the hyperbolic metric λ on the foliated ring S is the collection of complete hyperbolic metrics λ_i on the components of the critical decomposition of S . For example if $\cup_i S_i$ is the critical decomposition of S , then the space $HD(S)$ of harmonic differentials on S consists of the elements

$$\lambda^{-2} \bar{\phi} = \sum_i \frac{\bar{c}_i \lambda_i^{-2}}{z^2},$$

where $\phi \in A(S)$.

In fact, the space $HD(S)$ is isomorphic to the dual space $A(S(R))^*$ by way of the Petersen inner product

$$\langle \psi, \phi \rangle = \iint_{S(R)} \lambda^{-2} \overline{\psi} \phi.$$

The space of Teichmüller differentials $td(S)$ consists of the elements $\phi = \sum_i c_i \frac{z}{\bar{z}} \frac{d\bar{z}}{dz}|_{S_i}$, where $\cup_i S_i$ is the critical decomposition of S .

Denote by $\Omega(R)$ the set

$$\overline{\mathbb{C} \setminus \{\cup_n R^{-n}(P(R))\}},$$

then R acts on $\Omega(R)$ as an unbranched autocovering.

Now let $Y \subset \overline{\mathbb{C}}$ be an open subset. Then, as above, $A(Y)$ denotes the space of holomorphic functions on Y absolutely integrable over Y and $B(Y)$ consists of holomorphic functions ϕ on Y with the following norm

$$\|\phi\| = \sup_{z \in Y} |\Lambda_Y^{-2} \phi|,$$

where Λ_Y is a metric so that its restriction to any component $D \subset Y$ satisfies

$$\Lambda_{Y|D} = \lambda_D,$$

where λ_D is Poincaré metric on D .

Lemma (Bers Duality Theorem). *Let $Y \subset \overline{\mathbb{C}}$ be an open subset. Then the spaces $A(Y)$ and $B(Y)$ are dual by the Peterson scalar product*

$$\iint_Y (\Lambda_Y)^{-2} \phi \overline{\psi}.$$

Proof. See [Kra].

Poincaré Θ -operator for rational maps. We construct this operator in a way which is reminiscent of the construction in the case of a Kleinian group.

1). Let $D \in \Omega(R)$ correspond to an attractive periodic domain. Let $S_D \in S(R)$ be corresponding riemann surface. Then the projection $P : L(D) \mapsto S_D$ is a holomorphic unbranched covering. Let $P^* : A(L(D)) \mapsto A(S_D)$ be the push-forward operator: locally in charts $P^*(\phi) = \sum (\phi dz^2) \circ r_i$, where summation taken over all branches r_i of P . Then we call P^* – *Poincaré operator* for the attractive domain D and denote it by $\Theta_{L(D)}$.

2) The case of parabolic domains D is similar to that of an attractive domain.

3) Foliated case. This case corresponds to non-discrete groups. We need additional information pertaining to the foliated case. Let us start with a simple lemma about the Ruelle operator.

Lemma 1. *Let R be a rational map, $Y \subset \overline{\mathbb{C}}$ a positive Lebesgue measure subset which is completely invariant under the action of R . Then the following is true.*

(1) $R^* : L_1(Y) \rightarrow L_1(Y)$ is a linear surjection with unit norm. The operator

$$R_*(\phi) = \frac{\phi(R)(R')^2}{\deg(R)}$$

- is an isometric inclusion "into" and $R^* \circ R_* = I$, where I is the identity operator.
- (2) The Beltrami operator

$$B_R(\phi) = \phi(R) \frac{\overline{R'}}{R'} : L_\infty(Y) \rightarrow L_\infty(Y)$$

- is the dual operator to R^* . The operator B_R is an isometric inclusion.
- (3) If Y is an open set, then $R^* : A(Y) \rightarrow A(Y)$ is a surjection of unit norm and R_* maps $A(Y)$ into itself as well.

Proof. All items are immediate consequences of the definition of the operators.

Remark 2. Suppose $R : X \rightarrow Y$ is a branched covering with a rational map R and two domains $X, Y \subset \overline{\mathbb{C}}$. Then $R^* : A(X) \rightarrow A(Y)$ is the Poincaré operator of the covering R .

Now let us move onto the foliated case. For simplicity let D be an invariant either superattractive domain or Siegel disk or Herman ring. Our aim is to prove the following theorem.

Theorem 3. Let $D \subset F(R)$ be an invariant domain corresponding to the foliated case and $X = D \setminus P(R)$. Let S be the foliated surface associated with D . Then there exists a continuous linear projection $P^* : A(X) \mapsto A(S)$. The dual operator $P_* : HD(S) \mapsto HD(X)$ is an injection.

Proof. Let $Y \subset X$ be a component of critical decomposition and $O(Y)$ be its full orbit, then the projection $P : O(Y) \mapsto Y$ allows to construct the push-forward and pull-back maps. But in this situation the push-forward map maps $A(O(Y))$ onto the space $A(Y)$ of all integrable holomorphic quadratic differentials on Y . We need to consider average with respect to the group $G(D)$.

We claim

Claim. There exists a continuous linear projection $\pi : A(X) \mapsto A(S)$. The dual map π_* is an injection from $HD(S)$ into the space of Beltrami differentials on X .

Firstly assume that claim is proved, then we finish our proposition by setting $P^* = \pi \circ \{\text{push-forward map}\}$ and $P_* = \{\text{pull-back map}\} \circ \pi_*$.

In what follows we need the following basic facts about non-expansive operators and rational quadratic differentials.

Mean ergodicity lemma. Let T be a non-expansive ($\|T\| \leq 1$) linear endomorphism of a Banach space B and let $\phi \in B$ be any element.

- (1) Assume that for the Cesaro average $A_N(T, \phi) = \frac{1}{N} \sum_{i=0}^{N-1} T^i(\phi)$ there exists a subsequence $\{n_i\}$ such that $A_{n_i}(T, \phi)$ weakly converges to an element $f \in B$. Then f is a fixed point for T and $A_N(\phi)$ converges to f strongly (i.e. in norm). If $f = 0$ then $\phi \in \overline{(I - T)(B)}$ and vice versa i.e. if $\phi \in \overline{(I - T)(B)}$, then $A_n(T, \phi)$ tends to zero with respect to the norm.

- (2) A linear continuous operator T on a norm space B is called mean ergodic if and only if the Cesaro average $A_N(T, \phi)$ converges with respect to the norm for any element $\phi \in B$. In this case $B = \text{Fix}(T) \times \overline{(I - T)(B)}$ and $A_n(T)$ converges (in the strong topology) to the continuous projection $\pi : B \rightarrow \text{Fix}(T)$, where $\text{Fix}(T)$ is the space of fixed elements for T .
- (3) A linear non-expansive endomorphism T of a Banach space B is mean ergodic if and only if for any $\omega \in \text{Fix}(T^*)$ there exists $\phi \in \text{Fix}(T)$ so that $\omega(\phi) \neq 0$, where $\text{Fix}(T^*)$ is the space of fixed elements for the dual operator T^* .

Proof. See text of U. Krengel ([Kren]), theorem 1.1, page 72 and the theorem 1.3, page 73.

Lemma(Bers Density Theorem). Let C be a closed subset of $\overline{\mathbb{C}}$ and C_0 a dense subset of C . If Y is the complement to C in $\overline{\mathbb{C}}$, then let $A(Y) \subset L_1(\overline{\mathbb{C}})$ be subspace of the functions holomorphic on Y . Let $R(C_0) \subset L_1(\overline{\mathbb{C}})$ be subspace of the rational functions holomorphic outside of C_0 . Then $R(C_0)$ is everywhere dense subspace in $A(Y)$ in the L_1 -norm.

Proof. See the text of F.P. Gardiner and N. Lakic [GL].

Let Z be the surface Y equipped with the group $G(D)$. Any element $h \in A(Z)$ is invariant with respect to the group of all rotation of Y and vice versa. Then the average with respect to any dense subgroup $G \in S^1$ must produces elements of $A(Z)$.

Let us consider $g(z) = \rho z, \rho = \exp(2\pi i \alpha)$, with an irrational α . Then g defines an non-expansive operator $T : A(Y) \mapsto A(Y)$, by the formula $T(h)(z) = h(\rho z)\rho^2$ and the space $\text{Fix}(T) = A(Z)$.

Our aim is to show that $T : A(Y) \mapsto A(Y)$ is mean ergodic.

Let $W \subset A(Y)$ be the linear span of quadratic integrable rational differentials on $\overline{\mathbb{C}}$ with poles in $\overline{\mathbb{C}} \setminus \overline{Y}$. Then by the Bers's density theorem W is everywhere dense subset of $A(Y)$. If the Cesaro averages $A_n(T, h)$ converges in the strong topology for any element $h \in W$, then T is mean ergodic such as the Cesaro averages form equicontinuous family of operators.

Hence by above it is enough to show the strong convergence of $A_n(T, h)$ for any $h \in W$. Let $h \in Y$ be an element, then by assumptions h is holomorphic in a ring E which compactly contains the ring Y . Let $h(z) = \sum_{-\infty}^{+\infty} a_i z^i$ be the Laurent series of h , then the decomposition of $A_n(T, h)$ is as follows

$$\sum_{-\infty}^{+\infty} a_i z^i \frac{\sum_{l=0}^{n-1} \rho^{l(i+2)}}{n}$$

Such as $\rho = \exp(2\pi i \alpha) \neq 1$ and the series $\sum_{-\infty}^{+\infty} a_i z^i$ converges absolutely on \overline{Y} , then the series for $A_n(T, h)$ converges absolutely too on \overline{Y} . Hence the calculations by elements show that $\lim_{n \rightarrow \infty} \frac{\sum_{l=0}^{n-1} \lambda^{l(i+2)}}{n} = 0$, for any $i \neq -2$. as result we have

$$\lim_{n \rightarrow \infty} A_n(T, h) = \frac{a_{-2}}{z^2}$$

The uniform convergence in limit above on \overline{Y} implies the strong convergence in $A(Y)$.

The dual map $\pi_* : HD(Z) \mapsto HD(Y)$ is simply inclusion so that the following is true

$$\iint_Y \pi(h)\mu = \iint_Y h\pi_*(\mu),$$

for any $h \in A(Y)$ and $\mu \in HD(Y)$. Hence the claim and the theorem are proved.

Finally we define $\Theta(R) : A(\Omega) \rightarrow A(S_R)$ by

$$\Theta(R)(\phi) = (\Theta_{O(D_1)}, \dots, \Theta_{O(D_k)}) ,$$

where $D_i \subset F(R)$ are periodic components. If D_i is a superattractive or Siegel or Herman, then $\Theta_{O(D_i)} = P^*$. Here $O(D_i)$ means the full orbit of D_i .

The Space $A(R)$

Now again consider the space $A(\Omega)$. Note that any function of the form

$$\gamma_a(z) = \frac{a(a-1)}{z(z-1)(z-a)} \text{ for } a \in \overline{\mathbb{C}} \setminus \Omega$$

belongs to $A(\Omega)$. Let us introduce the subspace $A(R) \subset A(\Omega)$ as follows. Let S be the set

$$\{\cup_i \{O(c_i)\} \cup \{\{O(0, 1, \infty)\} \setminus \{0, 1, \infty\}\},$$

where c_i are the critical points . Then we set

$$A(R) = \text{linear span}\{\gamma_a(z), a \in S\}$$

This space $A(R)$ is a linear space and we introduce on $A(R)$ two different topologies though the norms $|\cdot|_1 = \int_{\Omega} |\cdot|$ and $|\cdot|_2 = \int_{J(R)} |\cdot|$. Denote by A_i the spaces $\{A(R), |\cdot|_i\}$, respectively.

Remark 4. *The space $A(R)$ serves as a kind of connection between the spaces $L_1(\Omega)$ and $L_1(J(R))$. The comparison of the $\|\cdot\|_1$ and $\|\cdot\|_2$ topologies is the basis for our discussion below.*

Lemma 5. *The operators R^* and R_* are continuous endomorphisms of A_1 and A_2 .*

Proof. It is sufficient to show that for any $\phi \in A(R)$ the functions $R^*(\phi)$ and $R_*(\phi)$ belong to $A(R)$.

Let $\phi = \gamma_a$. Then $R^*(\phi)$ and $R_*(\phi)$ are holomorphic everywhere except a finite number of points belonging to the set S and hence are rational functions holomorphic on Ω . Moreover both $R^*(\phi)$ and $R_*(\phi)$ are integrable over $\overline{\mathbb{C}}$ and hence belong to $A(R)$. This proves the lemma.

Lemma 6. *Let L be a continuous functional on A_1 invariant under the action of R^* (i.e. $L((R^*)(\phi)) = L(\phi)$). Then $L(\phi) = \iint_{\Omega} \lambda^{-2} \overline{\psi} \phi$, where λ is hyperbolic metric on Ω and $\psi \in B(\Omega)$.*

Proof. The Bers density and Bers duality theorems prove this lemma.

Now we need the following basic facts from the Ergodic Theory:

Let $T : X \mapsto X$ be measurable non-singular map of a measurable subset $X \subset \mathbb{C}$, with respect to the Lebesgue measure m .

Let $C \cup D = X$ be the Hopf decomposition of X onto conservative part C and dissipative part D . Let us recall that a measurable subset $W \subset X$ is *wandering* iff $m(T^{-n}(W) \cap T^{-k}(W)) = 0$, for any $k \neq n$. Then $D = \{\cup W, W \subset X \text{ is wandering}\}$, and $C = X \setminus D$. Since $T^{-1}(W)$ is wandering for wandering W , we have that $T^{-1}(D) \subset D$ modulo the Lebesgue measure. Hence $T(C) \subset C$ modulo the Lebesgue measure.

The map $T : X \mapsto X$ is called *conservative* if $m(D) = 0$. If $m(C) > 0$, then from above we have that $T|_C : C \mapsto C$ is a conservative map.

Poincaré Recurrence Theorem. *Suppose $T : X \mapsto X$ is a conservative, non-singular map. If (Z, d) is a separable metric space, and $f : X \mapsto Z$ is a measurable map, then*

$$\liminf_{n \rightarrow \infty} d(f(x), f(T^n(x))) = 0 \text{ for almost every } x \in X.$$

Proof. See text of J. Aaronson [Aa] page 17.

Definition. *A rational map R is ergodic if any measurable set A satisfying $R^{-1}(A) = A$ has zero or full measure in the sphere.*

Then the Poincaré Recurrence Theorem above and a result of Lyubich [L] (or see theorem "Attracting or ergodic" p.42 in the book of C. McMullen ([MM1])) give the following alternative.

Lemma 7. *Let R be any rational map. Let C be the conservative part of the Hopf decomposition of $J(R)$. Assume that the Lebesgue measure $m(C) > 0$, then*

- (1) $C \subset P(R)$ modulo the Lebesgue measure, or
- (2) the Julia set is equal to the whole Riemann sphere and the action of R is ergodic.

Proof. Let $B = C \setminus P(R)$. Assume $m(B) > 0$. Then by the Poincaré Recurrence Theorem with $(Z, d) = (\overline{\mathbb{C}}, d)$, where d is the spherical metric and $f = id$, we obtain that

$$\limsup d(R^n(x), P(R)) > 0,$$

for almost every $x \in B$. Hence by the arguments of theorem of McMullen which are mention above the map R is ergodic. We complete this lemma.

Now we ready to prove the theorem A.

Theorem A. *Let $R \in X$ be a rational map. Then either R is a Lattés map or there is no regular fixed points for $R^* : L_1(\overline{\mathbb{C}}) \mapsto L_1(\overline{\mathbb{C}})$.*

Proof. Assume $\phi \neq 0 \in L_1(\overline{\mathbb{C}})$ is a fixed regular point for R^* . Then the dual operator $B_R : L_\infty(\overline{\mathbb{C}}) \mapsto L_\infty(\overline{\mathbb{C}})$ has a fixed point μ so that $\iint_{\overline{\mathbb{C}}} \mu \phi \neq 0$. The lemma 7 above and theorem 3.17 (Toral or attracting) in [MM1] imply either R is a flexible Latté map or the complement to the postcritical set $\{\overline{\mathbb{C}} \setminus P(R)\} \subset D$, where D is the dissipative set for the map R .

Then $|\phi| = 0$ almost everywhere on D and hence supporter of the complex valued measure $\overline{\partial}\phi$ belong to $P(R)$.

Now we can assume that $P(R)$ is a compact subset of the plane. Otherwise if $P(R)$ is unbounded let $h(z)$ be a Mobius map which maps $P(R)$ into the plane, then the differential $\phi_1 = \phi(h)(h')^2$ is invariant for the map $R_1 = h \circ R \circ h^{-1}$ and easy calculations show that $\overline{\partial}\phi_1$ is a complex valued finite measure.

Let $V(z) = \iint_{\mathbb{C}} \frac{\overline{\partial}\phi(\xi)}{\xi - z}$ be the Cauchy transform of the measure $\overline{\partial}\phi$. Easy calculations show that in distributions $\overline{\partial}V = \overline{\partial}\phi$ and $V(z)$ is holomorphic out of $P(R)$, and $V(z) \mapsto 0$, as $z \mapsto \infty$. Then we claim:

Claim. $\phi = V$ almost everywhere on the plane.

Proof of the claim. Indeed $\overline{\partial}(\phi - V) = 0$ in distribution, hence the function $\phi - V$ is holomorphic on the plane. Besides $\phi = 0$ out of $P(R)$ and $\phi(z) - V(z) \mapsto 0$, as $z \mapsto \infty$ hence $\phi = V$ almost everywhere.

To finish the proof of theorem A we need

Proposition 8. *Let Z be a compact subset of the plane with empty interior and $\omega \neq 0$ be a complex-valued finite measure on Z . Let $V(z) = \iint_{\mathbb{C}} \frac{\omega(\xi)}{\xi - z}$ be Cauchy transform. Then the function $V(z)$ is not identically zero on $Y = \mathbb{C} \setminus Z$ in the following cases*

- (1) *the set Z has zero Lebesgue measure,*
- (2) *if the diameters of the components of $\mathbb{C} \setminus Z$ are uniformly bounded from below away from zero or*
- (3) *If O_j denote the components of Y , then $Z \in \cup_j \partial O_j$.*

Proof. The first is evident.

2) Assume that $V = 0$ identically outside of Z . Let $R(Z) \subset C(Z)$ denote the algebra of all uniform limits of rational functions with poles outside of Z in the sup-topology. Here $C(Z)$ denotes as usual the space of all continuous functions on Z with the sup-norm. Then the measure ω induces a linear functional on $R(Z)$. Items (2) and (3) are based on the generalized Mergelyan theorem (see [Gam] thm. 10.4) which states that *If diameters of all components of $\mathbb{C} \setminus Z$ are bounded uniformly from below away from 0, then every continuous function holomorphic on the interior of Z belongs to $R(Z)$.*

Let us show that ω annihilates the space $R(Z)$. Indeed, let $r(z) \in R(Z)$ be a rational function and γ a curve enclosing Z close enough to Z such that $r(z)$ does not have poles in

the interior of γ . Then since $l = 0$ outside of Z we need only apply Fubini's theorem:

$$\int r(z) d\omega(z) = \int d\omega(z) \frac{1}{2\pi i} \int_{\gamma} \frac{r(\xi) d\xi}{\xi - z} = \frac{1}{2\pi i} \int_{\gamma} r(\xi) d\xi \int \frac{d\omega(z)}{\xi - z} = \frac{1}{2\pi i} \int_{\gamma} r(\xi) V(\xi) d\xi = 0.$$

By the generalized Mergelyan theorem, we have $R(Z) = C(Z)$ and $\omega = 0$, contradiction.

Now let us check (3). We **claim** that $V = 0$ almost everywhere on $\cup_i \partial O_i$.

Proof of the claim. Let $E \subset \cup_i \partial O_i$ be any measurable subset with positive Lebesgue measure. Then the function $F_E(z) = \iint_E \frac{dm(\xi)}{\xi - z}$ is continuous on $\mathbb{C} \setminus \cup_i O_i$ and is holomorphic on the interior of $\mathbb{C} \setminus \cup_i O_i$. Again, by the generalized Mergelyan theorem $F_E(z)$ can be approximated on $\mathbb{C} \setminus \cup_i O_i$ by functions from $R(\mathbb{C} \setminus \cup_i O_i)$; and hence by the above arguments and by our hypotheses we have $\int F_E(z) d\omega(z) = 0$. But another application of Fubini's theorem gives

$$0 = \int F_E(z) d\omega(z) = \int d\omega(z) \iint_E \frac{dm(\xi)}{\xi - z} = \iint_E dm(\xi) \int d\omega(z) \frac{1}{\xi - z} = \iint_E V(\xi) dm(\xi).$$

Hence for any measurable $E \subset \cup_i \partial O_i$, we have $\iint_E V(z) = 0$. This proves the claim.

Now for any component $O \in Y$ and any measurable $E \subset \partial O$ we have $\iint_E V(z) = 0$. By assumption $V = 0$ almost everywhere on \mathbb{C} , contradiction with $\omega \neq 0$. This finishes the proof of the proposition.

To finish the theorem A in we only need to note that under our assumption the postcritical set $P(R)$ has empty interior.

Bers Isomorphism

Here we reproduce the Bers construction for Beltrami differentials and Eichler cohomology with corrections (which are often obvious) for the rational maps.

Consider the Beltrami action of R on the space $L_{\infty}(\overline{\mathbb{C}})$ i.e.

$$B_R(\phi)(z) = \phi(R)(z) \frac{\overline{R'(z)}}{R'(z)}.$$

Thus the subspace Fix of fixed points for B_R in $L_{\infty}(\overline{\mathbb{C}})$ is in fact the space of invariant Beltrami differentials for R . The unit ball in this space describes all quasiconformal deformations of R .

Now normalize R so that $0, 1, \infty$ are fixed points for R . Let $K(R)$ be the component of the subset of rational maps in $\mathbb{C}P^{2d+1}$ fixing the points $0, 1$ and ∞ containing R .

Let $\mu \in Fix(R)$; then for any λ with $|\lambda| < \frac{1}{\|\mu\|}$, the element $\mu_{\lambda} = \lambda\mu \in B$. Let f_{λ} be the family of qc-maps corresponding to the Beltrami differentials μ_{λ} with $f_{\lambda}(0, 1, \infty) = (0, 1, \infty)$. Then the map

$$\lambda \rightarrow R_{\lambda} = f_{\lambda} \circ R \circ f_{\lambda}^{-1} \in K(R)$$

is a conformal map. Let $R_\lambda(z) = R(z) + \lambda G_\mu(z) + \dots$. Differentiation with respect to λ at the point $\lambda = 0$ gives the following equation

$$F_\mu(R(z)) - R'(z)F_\mu(z) = G_\mu(z),$$

where $F_\mu(z) = \frac{\partial}{\partial \lambda} f_\lambda(z)|_{\lambda=0}$ and $G_\mu(z) = \frac{\partial}{\partial \lambda} R_\lambda(z)|_{\lambda=0} \in H^1(R)$.

By the theory of qc maps (see for example [Krush]). For any $\mu \in L_\infty(\mathbb{C})$ and t with $|t| < \epsilon$ and ϵ small, there exists the following formula for the qc-map $f_{t\mu}$ fixing $0, 1, \infty$:

$$f_{t\mu}(z) = z - \frac{z(z-1)}{\pi} \iint_{\mathbb{C}} \frac{t\mu}{\xi(\xi-1)(\xi-z)} + |t|O(C(\epsilon, R)\|\mu\|_\infty^2),$$

where $|z| < R$ and $C(\epsilon, R)$ is a constant which does not depend on μ . In addition,

$$F_\mu(z) = \frac{\partial f_\lambda}{\partial \lambda}|_{\lambda=0} = -\frac{z(z-1)}{\pi} \iint_{\mathbb{C}} \frac{\mu}{\xi(\xi-1)(\xi-z)}.$$

By $H^1(R)$ we mean the complex tangent space to $K(R)$ at the point R . Then $H^1(R)$ may be described as follows: if $R(z) = z \frac{P_0}{Q_0}$, then

$$H^1(R) = \{z \frac{PQ_0 - QP_0}{Q_0^2}, \text{ where } Q(1) = P(1), \deg(Q) \leq \deg(R), \deg(P) \leq \deg(R) - 1\},$$

In the above P, Q are polynomials and $\dim(H^1(R)) = 2d - 2$.

Remark 9. We use the notation $H^1(R)$ for the following reasons:

- (1) The Weyl cohomology' construction for the action of R (by the formula $\tilde{R}(f) = \frac{f(R)}{R'}$) on the space of all rational functions gives the space H which is isomorphic to the tangent space to $\mathbb{C}P^\infty$ at R (up to normalization). More precisely H is equivalent to the direct limit

$$H^1(R) \xrightarrow{j_1} H^1(R^2) \xrightarrow{j_2} H^1(R^3) \dots,$$

where j_i are equivalent to the action \tilde{R} .

- (2) This construction for a Kleinian group gives Eichler cohomology.

Hence we can define a linear map $\beta : \text{Fix}(R) \rightarrow H^1(R)$ by the formula

$$\beta(\mu) = F_\mu(R(z)) - R'(z)F_\mu(z).$$

In analogy with Kleinian groups we call the map β the *Bers map* (see for example [Kra]).

Let $A(S(R))$ be the space of holomorphic integrable quadratic differentials on the disconnected surface $S(R)$. Let $HD(S(R))$ be the space of harmonic differentials on $S(R)$: these are differentials which in a local charts have the form $\alpha = \frac{\bar{\phi} dz^2}{\rho^2 |dz|^2}$, where $\phi dz^2 \in A(S(R))$

and $\rho|dz|$ is the Poincaré metric. In fact, the space $HD(S(R))$ is isomorphic to the dual space $A(S(R))^*$ by way of the Petersen inner product

$$\langle \psi, \phi \rangle = \iint_{S(R)} \rho^{-2} \overline{\psi} \phi.$$

Let $\Theta^* : A^*(S_R) \rightarrow A^*(\Omega)$ be the dual operator. Then the image $HD(R) = \Theta^*(A^*(S_R))$ is called *the space of harmonic differentials* and $\dim(HD(R)) = \dim(A^*(S_R)) = \dim(A(S_R))$. ■

By duality we have

$$\iint_{\Omega(R)} \Theta^*(\alpha) \phi = \iint_{S(R)} \alpha \Theta(\phi),$$

for any $\alpha \in HD(S(R))$ and $\phi \in A(\Omega(R))$. Thus the element $\beta = \Theta^*(\alpha)$ presents the trivial functional on the space $A(\Omega(R))$ if and only if $\alpha = 0$.

Let us recall that $T(J(R)) = \text{Fix}(B_R)|_{J(R)}$ is the space of invariant Beltrami differentials supported on the Julia set.

The following theorem is proved in [Mak2] and for convenience of the readers we reproduce the proof of this theorem.

Theorem 10. *Let R be a rational map. Then:*

- (1) β is an injection when restricted to $HD(R) \times T(J(R))$,
- (2) if R is structurally stable, then $\beta : HD(R) \times T(J(R)) \rightarrow H^1(R)$ is an isomorphism.

Proof. 1). Let $A(J(R)) \subset L_1(\overline{\mathbb{C}})$ be the subspace of functions which are holomorphic on $F(R)$. Then $A(J(R))$ is a Banach space with the L_1 -norm. Furthermore, let $A(R) \subset A(J(R))$ be the subspace of rational functions. In other words, $A(R)$ is the linear span of the functions $\gamma_a(z) = \frac{a(a-1)}{z(z-1)(z-a)}$, where $a \in J(R)$. Then by the Bers Density Theorem $A(R)$ is an everywhere dense subspace of $A(J(R))$.

Now let $\mu \neq 0 \in \ker(\beta) \cap \{HD(R) \times T(J(R))\}$; then we have

$$F_\mu(R(z)) = R'(z)F_\mu(z)$$

and hence $F_\mu = 0$ on the set of all non-parabolic periodic points, and hence $= 0$ on the Julia set as well. Now if $F(R) = \emptyset$, then $F_\mu = 0$ identically on $\overline{\mathbb{C}}$; and using the fact that $\mu = \overline{\partial}F_\mu$ (in the sense of distributions) we have $\mu = 0$.

If $F(R) \neq \emptyset$, then on $A(J(R))$ the functional $L_\mu(\phi) = \iint \mu \phi$, satisfies $L_\mu(\gamma_a(z)) = F_\mu(a) = 0$, for any $a \in J(R)$. By the Bers density theorem we have $L_\mu = 0$ on $A(J(R))$, hence $\mu = 0$ almost everywhere on $J(R)$. Hence we obtain $\mu \in HD(R)$ represents the zero - functional on the space $A(\Omega(R)) \subset A(J(R))$. By the discussion above, we have $\mu = 0$.

2) If R is structurally stable, then $\dim(HD(R) \times J_R) = \dim(H^1(R)) = 2\deg(R) - 2$. By 1) the operator β is linear and injective, hence an isomorphism.

Now, let all critical points c_i be simple. Then there exists a decomposition $\frac{1}{R'(z)} = \omega + \sum \frac{b_i}{z - c_i}$, where $\omega = \frac{1}{R'(\infty)}$ is the multiplier of ∞ and c_i are the critical points (by the residue theorem $b_i = \frac{1}{R''(c_i)}$). For $i = 1, \dots, 2\deg(R) - 2$ let $h_i(z) = \frac{1}{R'(z)} - \frac{b_i}{z - c_i}$.

Proposition 11. *For any rational map R with simple critical points, the following statements hold,*

(1) *Let $\gamma_a(z) = \frac{a(a-1)}{z(z-1)(z-a)} \in L_1(\overline{\mathbb{C}})$ where $a \in \mathbb{C} \setminus \{0, 1\}$ is not a critical point. Then*

$$R^*(\gamma_a(z)) = \frac{\gamma_{R(a)}(z)}{R'(a)} + \sum_i b_i \gamma_a(c_i) \gamma_{R(c_i)}(z).$$

Let $\tau_a(z) = \frac{1}{z-a}$, where $a \in \mathbb{C}$ is not a critical point. Then

$$R^*(\tau_a(z)) = \frac{\tau_{R(a)}(z)}{R'(a)} + \sum_i b_i \tau_a(c_i) \tau_{R(c_i)}(z).$$

(2) *If $a = c_i$ is a critical point, then*

$$R^*(\gamma_a(z)) = (h_i(a) + b_i \frac{2c_i - 1}{c_i(c_i - 1)}) \gamma_{R(a)}(z) + \sum_{j \neq i} b_j \gamma_a(c_j) \gamma_{R(c_j)}(z),$$

and

$$R^*(\tau_a(z)) = h_i(a) \tau_{R(a)}(z) + \sum_{j \neq i} b_j \tau_a(c_j) \tau_{R(c_j)}(z),$$

$$\text{where } h_i(a) + b_i \frac{2c_i - 1}{c_i(c_i - 1)} = \lim_{a \rightarrow c_i} \left(\frac{1}{R'(a)} + b_i \gamma_a(c_i) \right).$$

Proof. See lemma 5 in [Mak2]

From the proposition 11 we have

$$(*) \quad \beta(\mu)(z) = F_\mu(R(z)) - R'(z) \cdot F_\mu(z) = -R'(a) \sum_i b_i F_\mu(R(c_i)) \gamma_a(c_i).$$

Remark 12. *Proposition 11 gives a different set of coordinates for the spaces $H^1(R)$ and $HD(R) \times J_R$. Namely the formula * above describes the isomorphism $\beta^* : HD(R) \times J_R \rightarrow \mathbb{C}^{(2deg(R)-2)}$ by*

$$\beta^*(\mu) = (F_\mu(R(c_1)), \dots, F_\mu(R(c_{2deg(R)-2})))$$

Beltrami Differentials on Julia set

Here we discuss the space $T(J(R))$. Each element $\mu \in T(J(R))$ defines an *invariant* with respect to the Ruelle operator functional L_μ on the space $A(R)$, which is continuous in the topology of A_2 (recall that $A_i = (A(R), |\cdot|_i)$). Continuity of L_μ in the topology of the space A_1 is crucial as regards the question of non-triviality of μ . Indeed we have the following lemma.

Lemma 13. *Let $\mu \in T(J(R))$, then $\mu = 0$ if and only if L_μ is a continuous functional on A_1 .*

Proof. Let L_μ be continuous on A_1 . Then by the Bers' density theorem L_μ is continuous on $A(\Omega)$. By lemma 9, there exists an element $\psi \in A(S_R)$, such that the functional $L_\mu(\alpha) = \iint_{\mathbb{C}} \alpha \lambda^{-2} \overline{\psi}$, and hence $F_\mu(a) = L_\mu(\gamma_a) = \iint_{\mathbb{C}} \gamma_a \lambda^{-2} \overline{\psi} = F_{\lambda^{-2} \overline{\psi}}(a)$. Hence $\beta(\mu)(a) = \beta(\lambda^{-2} \overline{\psi})(a)$ for any $a \in S$. The set S is an infinite subset of the plane, hence the rational functions $\beta(\mu)$ and $\beta(\lambda^{-2} \overline{\psi})$ are equal. This contradicts to the injectivity of β , and the lemma is proved.

Now we begin to consider the relationship between the continuity of L_μ for $\mu \in \mathbf{J}_R$ and certain properties of the Ruelle operator $R^* : A_2 \rightarrow A_2$. Recall that the operator R^* acts as a linear endomorphism of $L_1(\mathbf{J}(R))$ with unit norm.

Proposition 14. *Let $R \in X$ be a rational map with simple critical points. Assume that R is not the Lattés map. Then*

- (1) *$T(J(R)) = \emptyset$ if and only if the Ruelle operator $R^* : A_2 \rightarrow A_2$ is mean ergodic,*
- (2) *Assume in addition that $F(R) \neq \emptyset$, and $m(P(R)) = 0$, then $m(J(R)) = 0$, if and only if the modulus of the Ruelle operator $|R_*| : L_1(J(R)) \rightarrow L_1(J(R))$ is mean ergodic.*

Proof. (1). If $T(J(R)) = \emptyset$, then the subspace $(I - R^*)(A_2)$ is everywhere dense in A_2 , and by item (2) of the Mean ergodicity lemma we are done.

Now suppose that R^* is mean ergodic on A_2 . Let $\mu \neq 0 \in T(J(R))$, then there exists an element $\gamma \in A_2$ such that $\iint_{\mathbb{C}} \mu \gamma \neq 0$. Let $\gamma_n = A(n, R)(\gamma)$ be Cesaro averages, then by the Theorem A the limit $\lim_{n \rightarrow \infty} \gamma_n = 0$ in the strong topology on A_2 . Beside we have

$$\lim_{n \rightarrow \infty} \iint_{\mathbb{C}} \mu \gamma_n = \lim_{n \rightarrow \infty} \iint_{\mathbb{C}} \mu \gamma = \iint_{\mathbb{C}} \mu \gamma \neq 0.$$

The contradiction above complete the proof of the (1).

(2). If $m(J(R)) = 0$, then the space $L_1(J(R)) = \{0\}$, and we are done.

Now let $|R_*| : L_1(J(R)) \rightarrow L_1(J(R))$ be mean ergodic.

Assume that $m(J(R)) > 0$. Let $\phi > 0 \in L_1(J(R))$ be any element with $\iint_{J(R)} \phi = 1$. Then the mean ergodicity of $|R_*|$ implies that the Cezaro averages $A(N, |R_*|, \phi)$ converges to an element ψ so that

- (1) $|R_*|\psi = \psi$ and
- (2) $1 = \iint_{J(R)} \phi = \iint_{J(R)} A(N, |R_*|, \phi) = \iint_{J(R)} \psi$.

Now under assumption of the proposition and by the Lemma 7 above we have that $m(C) = 0$. Hence $J(R) = D$ modulo the Lebesgue measure. If $W \subset J(R)$ is wandering then $\iint_W \psi = 0$, and hence $\psi = 0$ on $D = J(R)$ which is contradiction with (2) above.

We will now show that the topologies $\|\cdot\|_1$ and $\|\cdot\|_2$ are "mutually disjoint." Denote by X_i the closure of the space $(I - R^*)(A(R))$ in the spaces A_1 and A_2 .

Proposition 15. *Let R be a rational map and $\dim(A(S_R)) \geq 1$. Then the following conditions are equivalent.*

- (1) *the map $i = \text{id} : A_1 \rightarrow A_2$ maps weakly convergent sequences onto weakly convergent sequences.*
- (2) *$i(X_1) \supset X_2$,*
- (3) *the Lebesgue measure of the Julia set is zero.*

Proof. Condition (3) trivially implies conditions (1) and (2).

Assume condition (1) holds. Then the dual map $i^* : A_2^* \rightarrow A_1^*$ is continuous in the $*$ -weak topologies on A_1^* and A_2^* . Hence for any $\mu \in A_2^* \subset L_\infty(J)$, there exists an element $\nu \in A_1^* \subset L_\infty(F)$ such that $\nu = i^*(\mu)$ and

$$\iint_J \mu \gamma = \iint_F \nu \gamma.$$

Then for any $\gamma \in A(R)$ we have $\iint_{\mathbb{C}} \gamma(\mu - i^*(\mu)) = 0$. Let $F_\mu(z)$ and $F_\nu(z)$ be potentials. Then $F|_{J(R)} = (F_\mu(z) - F_\nu(z))|_{J(R)} = 0$ and if $m(J(R)) > 0$ we have $F_{\bar{z}} = 0$ almost everywhere on $J(R)$, where $F_{\bar{z}}$ is defined in the sense of distributions. Hence we deduce:

$$\mu - i^*(\mu) = 0$$

almost everywhere on $J(R)$. Since $F(R) \cap J(R) = \emptyset$, we have $\mu = 0$ almost everywhere and we conclude that $A_2^* = \{0\}$. Hence $A_2 = \{0\}$, which gives $m(J(R)) = 0$.

Now assume (2). Then the hypothesis implies that any invariant continuous functional on A_1 generates an invariant line field on the Julia set contradicting the injectivity of the Bers map. By the assumption R always has non-trivial qc-deformation and hence, we conclude that $m(J(R)) = 0$.

Proposition 16. *Assume that $\dim(A(S_R)) \geq 1$ and $m(J(R)) > 0$ for the given rational map R . Then there exist no invariant line fields on the Julia set if and only if $i^{-1}(X_2) \supset X_1$.*

Proof. If there exist no invariant line fields, then $X_2 = A_2$. Now assume $i^{-1}(X_2) \supset X_1$; then existence of an invariant line field would contradict the injectivity of the Bers map.

We finish this chapter with the next theorem.

Theorem 17. *Let $R(z)$ be a rational map and $c \in J(R)$ be a critical point. Let $S_L = \sum_{j=0}^L \frac{1}{(R^j)'(R(c))}$. Assume that there exists a subsequence $\{n_i\}$ of integers such that the sequence $\{R^{n_i+1}(R(c))\}$ is bounded and either*

- (1) *$\lim_{i \rightarrow \infty} |(R^{n_i})'(R(c))| = \infty$ and $\overline{\lim}_{i \rightarrow \infty} |S_{n_i}| > 0$ or*
- (2) *$|(R^{n_i})'(R(c))| \sim C = \text{Const}$ for $i \rightarrow \infty$ and $\overline{\lim}_{i \rightarrow \infty} |S_{n_i}| = \infty$.*

Then R is not structurally stable (is an unstable map).

Proof. Consider the one-dimensional family of deformations $R_\lambda(z) = R(z) + \lambda$. Assume that R is stable. Then there exist an ϵ and a holomorphic family $h_\lambda : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ of qc-homeomorphisms such that for any $|\lambda| < \epsilon$,

$$R_\lambda(z) = h_\lambda \circ R \circ h_\lambda^{-1}(z).$$

Let $V(z) = \frac{\partial h_\lambda}{\partial \lambda}|_{\lambda=0}(z)$; then the derivative of the equation above with respect to λ evaluated in $\lambda = 0$ gives the equation

$$V(R(z)) = 1 + R'(z)V(z).$$

The function $V(z)$ is continuous on \mathbb{C} and for any critical point c we have $V(R(c)) = 1$.

Now for any m and $z \notin R^{-m}(\infty)$, we calculate

$$\frac{V(R^m(z))}{(R^m)'(z)} = V(z) + \frac{1}{R'(z)} + \frac{1}{(R^2)'(z)} + \dots + \frac{1}{(R^m)'(z)}.$$

Setting $z = R(c)$ and $m = n_i$, we obtain in both cases a contradiction.

Measures. Proof of theorem B.

Start again with a rational map R . Consider an element $\gamma \in A(R)$ and the corresponding Cesaro average sequence $A_N(R)(\gamma) = \frac{1}{N} \sum_{i=0}^{N-1} (R^*)^i(\gamma)$. Let $C(U)$ be the space of continuous functions defined on \overline{U} for a fixed essential neighborhood U . Then any $*$ -weak limit of $A_N(R)(\gamma)$ on $C(U)$ is called a *weak boundary* of γ respect to R^* over U ; denote the set of all limit measures by $\gamma(U, R)$.

Proposition 18. *Let R be a structurally stable rational map with non empty Fatou set. Assume there exists a non-zero weak boundary $\mu \in \gamma(U, R^*)$ for an element $\gamma \in A(R)$ and an essential neighborhood U . Then the Lebesgue measure $m(J(R)) > 0$ and there exists a non-trivial invariant line field on $J(R)$.*

Proof. Under the assumptions, there exists an essential U , $\gamma \in A(R)$ and a subsequence N_i such that

- (1) $\iint \phi A_{N_i}(R)(\gamma)$ converges for any $\phi \in C(U)$ and
- (2) there exists $\psi \in C(U)$ such that $\lim_{i \rightarrow \infty} \iint \psi A_{N_i}(R)(\gamma) \neq 0$.

By density of the space of compactly supported continuous function in the space $C(U)$, we may assume that ψ has compact support $D \subset \overline{U}$. Extending ψ to $\overline{\mathbb{C}} \setminus D$ by zero, we obtain $\lim_{i \rightarrow \infty} \iint_{\overline{\mathbb{C}}} \psi A_{N_i}(R)(\gamma) \neq 0$. Hence the dual average $A_N(B_R)(\psi) = \frac{1}{N} \sum_{i=0}^{N-1} (B_R)^i(\psi)$ has non-zero $*$ -weak limit element in the $*$ -weak topology on $L_\infty(J(R))$. Let $\mu \in L_\infty(J(R))$ be this non-zero limit element. Then μ is fixed for B_R and $\mu = 0$ on $F(R)$ by construction. Hence $m(J(R)) > 0$ and μ defines the desired invariant line field.

It is not clear if the converse is true. We suggest the following conjecture.

Conjecture. *Let R be a rational map with non-empty Fatou set. The $T(J(R)) = \emptyset$ if and only if the weak boundaries $\gamma(U, R^*) = 0$ for all $\gamma \in A(R)$ and every essential neighborhood U .*

In general the absence of invariant line fields on the Julia set implies mean ergodicity of R^* on $L_1(J(R))$, and so it would be interesting to understand the conditions implying mean ergodicity of R^* from the measure-theoretic point of view. To do this, let us recall the definition of the following objects:

- (1) U is an essential neighborhood of $J(R)$ and

- (2) $H(U)$ consists of $h \in C(\overline{U})$ such that $\frac{\partial h}{\partial \bar{z}}$ (in sense of distributions) belongs to $L_\infty(U)$
- (3) $H(U)$ inherits the topology of $C(\overline{U})$.

Measures ν_l^i .

- (1) Let c_i and d_i be critical points and critical values, respectively. Then define $\mu_n^i = \frac{\partial}{\partial \bar{z}}((R^*)^n(\gamma_{d_i}(z)))$ (in sense of distributions).
- (2) Define by ν_l^i the average $\frac{1}{l} \sum_{k=0}^{l-1} \mu_k^i$.

Proof of Theorem B. Suppose that $\nu_{l_k}^1$ converges in the $*$ -weak topology on $H(U)$ for a subsequence $\{l_k\}$ and an essential neighborhood U . Then the sequence of averages $A_n(R)(\gamma_{d_1}(z)) \in L_1(U)$ is weakly convergent. If $m(J(R)) > 0$, this means $A_n(R)(\gamma_{d_1}(z))$ converges strongly in $L_1(J(R))$. Let $f = \lim_{n \rightarrow \infty} A_n(R)(\gamma_{d_1}(z))$, then by the arguments of the theorem A either $f = 0$, or R is a Lattés. In the last case R is an instable map. Now let $\mu \in HD(R) \times T(J(R))$, then $F_\mu(d_1) = \iint_{\mathbb{C}} \mu \gamma_{d_1}(z) = \iint_{\mathbb{C}} \mu A(n, R)(\gamma_{d_1}(z)) = \lim_{n \rightarrow \infty} \iint_{\mathbb{C}} \mu A(n, R)(\gamma_{d_1}(z)) = 0$. Contradiction with injectivity of the Bers map.

Proof of the Corollary B. Assume that the measures $\nu_{l_k}^i$ converges in the $*$ -weak topology on $H(U)$ for all i , a subsequence $\{l_k\}$ and an essential neighborhood U . Then the $A_{l_k}(R)(\gamma_{d_i}(z))$ converge strongly in $L_1(J(R))$.

Now let $\mu \neq 0 \in T(J(R))$. If $d \in J(R)$ is a critical value, then by the arguments of the theorem C $F_\mu(d) = 0$. Hence $\beta(\mu) = 0$, and $\mu = 0$. The contradiction with assumption complete the proof.

Now assume $T(J(R)) = 0$. Let us show that $\nu_l^i \rightarrow 0$ in the $*$ -weak topology on $H(U)$ for any essential neighborhood U . Otherwise, there exists a sequence $\{l_k\}$, an essential neighborhood U and a function $F \in H(U)$ such that

$$\lim_{k \rightarrow \infty} \iint F \nu_{l_k}^{i_0} = \lim_{k \rightarrow \infty} \iint F_{\bar{z}} A_{l_k}(R)(\gamma_{d_{i_0}}(z)) \neq 0.$$

Hence by the Mean Ergodicity Lemma $\lim_{k \rightarrow \infty} A_{l_k}(R)(\gamma_{d_{i_0}}(z)) = \phi \neq 0$. Then R is a Lattés by the theorem A, and hence $T(J(R)) \neq 0$. The contradiction with assumption complete the proof of the Corollary B.

Proof Theorem C

We begin by collecting some facts (see books of I.Kra "Automorphic forms and Kleinian Group" I. N. Vekua "Generalized analytic function.")

Facts. Denote by $F_\mu(a)$ the following integral $\iint_{\mathbb{C}} \mu(z) \tau_a(z) dz d\bar{z}$ where $\tau_a(z) = \frac{1}{z-a}$ for $a \in \mathbb{C}$ and $\mu \in L_\infty(J(R))$. Then

- (1) $F_\mu(a)$ is a continuous function on \mathbb{C} and $\frac{\partial F_\mu(a)}{\partial \bar{z}} = \mu$ in the sense of distributions.
- (2) $|F_\mu(a)| = O(|z|^{-1})$ for large z . $\|F_\mu(a)\|_\infty \leq \|\mu\|_\infty M$, where M does not depend on μ and $a \in \mathbb{C}$.
- (3) $|F_\mu(a_1) - F_\mu(a_2)| \leq \|\mu\|_\infty C |a_1 - a_2| |\ln |a_1 - a_2||$, where C does not depend on μ and a .

Denote by $B : L_\infty(J(R)) \rightarrow C(\mathbb{C})$ the operator $\mu \rightarrow F_\mu(a)$ and by X the image $B(L_\infty(J(R)))$. Let W denote the space X with the following topology:

$$\phi_n \rightarrow 0 \text{ iff } \|\phi_n\|_\infty \rightarrow 0 \text{ and } \frac{\partial \phi_n}{\partial \bar{z}} \rightarrow 0 \text{ in the } *-weak \text{ topology of } L_\infty(J(R)).$$

Lemma 19.

- (1) W is a complete locally convex vector topological space.
- (2) B is a compact operator mapping $L_\infty(J(R))$ onto W .
- (3) Any bounded set $U \subset W$ is precompact.

Proof. Item (1) is obvious.

2). Let $U \subset L_\infty(J(R))$ be bounded. Then U is precompact in the $*$ -weak topology on $L_\infty(J(R))$. Furthermore, from item (2) of Facts, we have that $B(U)$ forms a uniformly bounded and equicontinuous family of continuous functions. This means $B(U)$ is precompact in the topology of uniform convergence.

3). Boundedness in W means in particular that the set

$$V = \left\{ \frac{\partial \phi}{\partial \bar{z}} \text{ in sense of distributions, for } \phi \in U \right\}$$

forms a bounded set in the $*$ -weak topology of $L_\infty(J(R))$. Hence V is bounded in the norm topology of $L_\infty(J(R))$. We finish the lemma by using item (2) and the fact that $\phi = B(\phi_{\bar{z}})$.

Define an operator T on X as follows

$$T(F_\mu(a)) = F_{B_R(\mu)}(a) = \iint_{\mathbb{C}} B_R(\mu) \tau_a = \iint_{\mathbb{C}} \mu R^*(\tau_a(z)).$$

By Proposition 11 we have

$$T(\phi) = \frac{\phi(R(a))}{R'(a)} - \sum \frac{b_i \phi(R(c_i))}{a - c_i},$$

where b_i are residue the function $\frac{1}{R'(a)}$ in the critical point c_i . For example for $R(z) = z^2 + c$ we have $T(\phi)(a) = \frac{\phi(R(a)) - \phi(c)}{R'(a)}$.

Remark 20. By definition it may be seen that

$$\{T^n(\phi), n = 0, 1, \dots\}$$

forms bounded set in W .

Lemma 21. *T is a continuous endomorphism of W .*

Proof. Let $F_{\mu_i} \rightarrow 0$ in W ; then $\|\mu_i\| \leq C < \infty$ and hence $\{T(F_{\mu_i})\}$ forms a precompact family in W . Let ψ_0 be a limit point of this set. Then

$$\psi_0(a) = \lim_j T(F_{\mu_{i_j}}) = \iint_{\mathbb{C}} \mu_{i_j} R^*(\tau_a) \rightarrow 0 (* - \text{weak topology}).$$

Thus $\psi_0 = 0$.

Now let P be a strongly convergent polynomial. Let $\Delta \subset \overline{\mathbb{C}}$ be a closed topological disk centered at infinity which does not contain critical point of any iteration P^n . Then the family of functions

$$s_n(z) = \sum \frac{|b_i^n|}{|z - c_i^n|}$$

is uniformly bounded on Δ , where $\sum \frac{b_i^n}{z - c_i^n} = \frac{1}{(P^n)'(z)}$.

Lemma 22. *Assume $s_n(a) \leq C < \infty$ for all n for a given polynomial P . Then the Cesaro average $A_N(\tau_a)$ converges with the $L_1(J)$ -norm.*

Proof. In the notations above, we have

$$\begin{aligned} T(F_\mu)(y) &= \iint_J \frac{B(\mu)}{z - y} dz \wedge d\bar{z} = \iint_J \mu R^*(\tau_y) dz \wedge d\bar{z} \\ &= \frac{F_\mu(P)(y)}{P'(y)} - \sum \frac{b_i F_\mu(R(c_i))}{y - c_i} = \sum \frac{b_i (F_\mu(R(y)) - F_\mu(R(c_i)))}{y - c_i}. \end{aligned}$$

Now consider the sequence of functionals $l_i(F) = (A_i(T)(F))(a)$ on W . Under assumption of the lemma, we have

$$|l_i(F)| \leq 2 \frac{1}{i} \sum_{j=0}^{i-1} s_j(a) \sup_{w \in \mathbb{C}} |F(w)|,$$

so the family of functionals $\{l_i\}$ can be extended onto the space $C(\overline{\mathbb{C}})$ of continuous functions on $\overline{\mathbb{C}}$ to the family of uniformly bounded functionals. Therefore we can choose a subsequence l_{i_j} converging pointwise to some continuous functional l_0 . Note that l_0 is the fixed point for the dual operator T^* acting on dual W^* . This means that the sequence $A_{i_j}(R^*)(\tau_a)$ weakly converges in $L_1(J)$, and hence by the Mean ergodicity Lemma, the whole sequence $A_N(\tau_a)$ converges in norm to a fixed element of R^* .

We are now ready to prove the theorem C.

Proof of Theorem C.

It is enough to show convergence of $A_N(R^*)(\frac{a(a-1)}{z(z-1)(z-a)})$ for any fixed $a \in S$.

Let us denote by Y the subset of elements from $L_1(J(P))$ on which the averages $A_N(P^*)$ are convergent. Note that Y is a closed space such that family $A_N(P^*)$ forms equicontinuous family of operators.

We claim that for any $a \in S$ the elements $\gamma_a(z)$ belong to Y .

Proof of the claim. Otherwise, there would exist a continuous functional L on $L_1(J(R))$ and $a_0 \in S$ so that $L(\gamma_{a_0}) \neq 0$ and $Y \subset \ker(L)$. Note that L is an invariant functional (i.e. $L(R^*(f)) = L(f)$) such that for any $f \in L_1(J(R))$, the element $f - R^*(f)$ belongs to Y . Let $\nu \in L_\infty(J(R))$ be the element corresponding to L ; then ν is a fixed for Beltrami operator B_P and hence the function $F_\nu(a) = \iint \nu \tau_a$ is a fixed point for the operator T i.e.

$$\frac{F_\nu(P(a))}{P'(a)} - \sum \frac{b_i F_\nu(d_i)}{a - c_i} = F_\nu(a).$$

Let $d \in \Delta$ be a point such that $P(d) \in \Delta$; then by the arguments above $F_\nu(d) = F_\nu(P(d)) = 0$. Therefore the rational function $\Phi(a) = \sum \frac{b_i F_\nu(d_i)}{a - c_i}$ has a too large number of zeros. This immediately implies $\Phi(a) \equiv 0$ and the function F_ν satisfies the equation

$$\frac{F_\nu(P(a))}{P'(a)} = F_\nu(a).$$

Finally we have that F_ν is zero on the set of all repulsive periodic points, hence on the Julia set, hence everywhere because F_ν is holomorphic on the Fatou set. Thus $0 \neq L(\gamma_{a_0}) = (a_0 - 1)F_\nu(0) - a_0 F_\nu(1) + F_\nu(a_0) = 0$, contradiction.

To complete the theorem C we need the following lemma:

Lemma 23. *Let $T(J(P)) \neq 0$ for a strongly convergent polynomial P . Then there exists a regular fixed element for the Ruelle operator R^* .*

Proof. Let $\mu \neq 0 \in F(J(R))$ be non-trivial Beltrami differential, then there exists $a \in \Delta$ such that $0 \neq F_\mu(a) = \iint \mu \gamma_a(z)$. Then $f = \lim_{n \rightarrow \infty} A_n(R^*)(\gamma_a(z)) \neq 0$ is a fixed point for the Ruelle operator. Let us show that the total variation of $\bar{\partial}f$ is bounded. For this is sufficient to show that the total variation of $A_n(P^*)(\gamma_a(z))$ is bounded independently on n . Indeed we have,

$$|\iint \phi \bar{\partial} P^*(\tau_a)| = |\iint \bar{\partial} \phi P^*(\tau_a)| = \left| \frac{\phi(P(z))}{P'(a)} - \sum \frac{b_i F_\nu(d_i)}{a - c_i} \right| \leq s_1(a) \|\phi\|,$$

where ϕ is any differentiable function. Hence by the induction we have desired result.

Now the contradiction with the theorem A complete proof of the theorem C.

We will now give sufficient conditions on polynomial to be strongly convergent. The conditions will be given in terms of the Poincaré series of the rational map. We begin with the following calculations.

Lemma 24. *Let R be a rational map with no critical relations and simple critical points. Let c be a critical point of R and $d \in (R^k)^{-1}(c)$ be any point for some fixed k . Then for any fixed m the coefficient b corresponding to the entry $\frac{1}{z-d}$ in expression $s_m(z)$ has the following type*

$$b = \frac{1}{(R^m)''(d)} = \frac{1}{(R''(c))(R^{m-k-1})'(R(c))((R^k)'(d))^2}.$$

Proof. By a residue calculations at the point d .

Let us recall the backward and forward Poincaré series for the given rational map R .

Definition. *Forward Poincaré series* $P(x, R)$

$$P(x, R) = \sum_{n=0}^{\infty} \frac{1}{|(R^n)'(R(x))|}.$$

Backward Poincaré series $S(x, R)$.

Let $|R^*| = R_{1,1}$ be the modulus of the Ruelle operator, then

$$S(x, R) = \sum_{n=1}^{\infty} |R^*|^n(\mathbf{1}_{\mathbf{C}})(x) = \sum_{n=1}^{\infty} \sum_{R^n(y)=x} \frac{1}{|(R^n)'(y)|^2}.$$

Proof of Proposition C1. Let us again consider the function $s_n(a) = \sum \frac{|b_i|}{|a - c_i|}$ and let $B_n = \sum |b_i|$. Then by lemma above we have.

$$B_n = \sum_{c \in Cr(R)} \frac{1}{|R''(c)|} \sum_{j=1}^{n-1} \frac{1}{|(R^{n-j-1})'(R(c))|} \sum_{R^j(y)=c} \frac{1}{|(R^j)'(y)|^2},$$

hence we have the following formal equality

$$\sum_{n=2} B_n = \sum_{c \in Cr(R)} \frac{1}{|R''(c)|} S(c, R) \otimes P(c, R)$$

where \otimes means Cauchy product of series.

Corollary C. *Under condition of the theorem C above assume that for any critical point c there exists a constant M_c so that*

$$\frac{1}{|(P^n)'|} \leq \frac{M_c}{n} \text{ and } |R^*|^n(\mathbf{1}_{\mathbf{C}})(x) \leq \frac{M_c}{n},$$

then P is a strongly convergent polynomial.

Prof. The statement follows from properties of Cauchy product; e.g. the Cauchy product of two harmonic series is divergent but has uniformly bounded elements. We emphasize that evidently there is no rational maps for which the forward Poincaré series is equivalent to harmonic series for any critical point.

Proof of Corollary C2.

Let R be a rational map and $c_i, i = 1, \dots, 2\deg(R) - 2$, and $d_i, i = 1, \dots, 2\deg(R) - 2$, be critical points and critical values, respectively, and let $z = \infty$ be a fixed point with multiplier λ . Then by induction we have.

$$\begin{aligned} \frac{1}{R'(z)} &= \lambda + \sum_i \frac{b_i}{z - c_i} = \lambda + \sum_i \frac{1}{R''(c_i)} \frac{1}{z - c_i}, \\ \frac{1}{(R^n)'(z)} &= \lambda^n + \sum_i \sum_{k=0}^{n-1} \left(\sum_{y \in R^{-k}(c_i)} \frac{1}{(R^n)''(y)} \frac{1}{z - y} \right). \end{aligned}$$

We finished the proof of the first equality with the following lemma.

Lemma 25. *For any $k < n$*

$$\begin{aligned} \sum_{y \in R^{-k}} \frac{1}{(R^n)''(y)} \frac{1}{a - y} &= \frac{1}{R''(c_i)} \frac{1}{(R^{n-k-1})'(d_i)} \sum_j \frac{(J'_j)^2(c_i)}{a - J_j(c_i)} = \\ &= \frac{1}{R''(c_i)} \frac{1}{(R^{n-k-1})'(d_i)} (R^*)^k(-\tau_a)(c_i), \end{aligned}$$

where the J_j are branches of R^{-k} , $\tau_a(z) = \frac{1}{z-a}$ and R^* is Ruelle operator.

Proof. Lemma 24 and the above equalities.

We now prove the second equality. By proposition 11, we calculate as follows:

$$\begin{aligned} (R^*)^0(\tau_a)(z) &= \tau_a(z), (R^*)(\tau_a)(z) = \frac{1}{R'(a)(z - R(a))} - \sum_i \frac{b_i}{(a - c_i)(z - R(c_i))} \\ (R^*)^2(\tau_a)(z) &= \frac{1}{(R^2)'(a)(z - R^2(a))} - \frac{1}{R'(a)} \sum_i \frac{b_i}{(R(a) - c_i)(z - R(c_i))} - \\ &\quad \sum_i \frac{b_i}{(a - c_i)} R^*\left(\frac{1}{(z - R(c_i))}\right) \end{aligned}$$

and by induction,

$$\begin{aligned} ((R^*)^n(\tau_a)(z) &= \frac{1}{(R^n)'(a)(z - R^n(a))} - \\ &- \sum_i b_i \left(\frac{1}{(R^{n-1})'(a)(R^{n-1}(a) - c_i)(z - R(c_i))} + \dots + \frac{1}{a - c_i} (R^{n-1})^* \left(\frac{1}{(z - R(c_i))} \right) \right). \end{aligned}$$

Then summation with respect to n gives the desired equality.

Few Open Questions on spectrum of Ruelle and Beltrami operators and so on

The next proposition is completely trivial and we are left proof to the reader.

Proposition 26. *The spectrum of the operator $R^* : L_1(\overline{\mathbb{C}}) \rightarrow L_1(\overline{\mathbb{C}})$ is the closed unit disk Δ . Any interior point $\lambda \in \Delta$ is the eigenvalue.*

Now let $Y = \text{linear span}\{\gamma_a(z), a \in P(R)\} \subset L_1(\overline{\mathbb{C}})$ and Z be closure of Y , then we have the following connection of the dynamic of R and spectrum of its Ruelle operator.

Proposition 27.

- (1) R^* maps Z into Z .
- (2) The spectrum of $R^* : Z \rightarrow Z$ is finite pure point spectrum if and only if R is postcritically finite map.

Proof. (1). This case follows immediately from the proposition 11.

(2) If R is a postcritically finite, then $\dim(Z) < \infty$ and we are done.

Assume That $P(R)$ is infinite set, then there exists a point $a \in P(R)$ with infinite forward orbit and hence $\gamma_n = R^*(\gamma_a(z))$ are different pairwise. Let $\lambda_i, i = 1, \dots, N$ be the constants so that $\gamma_a(z) = \sum_i \lambda_i \phi_i$, where ϕ_i are eigenfunctions. Now let $\alpha_i, i = 1, \dots, N$ be the points of the spectrum, here $\alpha_0 = 0$ if zero belongs to the spectrum. Then we have infinitely many equations:

$$\gamma_n = \sum_i \alpha^n \lambda_i \phi_i$$

Hence there exists an $i_0 \neq 0$ so that the eigenfunction ϕ_{i_0} is rational and indeed is a linear combination of finite number of γ_k . Moreover $R^*(\phi_{i_0}) = \alpha_{i_0} \phi_{i_0}$. If A be the set of the poles of ϕ_{i_0} , then $A \cap \{\cup_n R^n(a)\} \neq \emptyset$. By the proposition 7 we have $R^n(A) \subset A \cup \{\text{critical values of } R^n\}$ for any $n > 0$. Hence the forward orbit of a is finite which is a contradiction.

It is interesting describe the spectrum of the operator $R^* : Z \rightarrow Z$.

Is it true that spectrum (not pure pointed) is finite only in a case of postcritically finite map ?

Is it possible that for a rational map R the operator $R^* : Z \rightarrow Z$ is a compact operator with infinite spectrum?

Spectrum of Beltrami operator.

Let us recall that a rational map $R \in \text{Rat}_d$ is in *general position* iff the cardinality of its critical values $= 2d - 2$. Particularly that means that any two critical points of R have different images.

Definition. Let $R \in \text{Rat}_d$, then the Hurwitz class $H(R)$ of the map R is the following space

$$H(R) = \{g \in \text{Rat}_d, \text{ there exist homeomorphisms } \phi, \psi, R \circ \phi = \psi \circ g\}.$$

Note that the Hurwitz class was introduced in holomorphic dynamic by A.Eremenko and M.Lyubich [EL] for entire function. We call this set as a Hurwitz class because the Hurwitz theorem describes classes of branched coverings like this (see for example [B]).

Lemma 28.

- (1) Let R and Q be two rational maps in general position, then $Q \in H(R)$ if and only if $\deg(R) = \deg(Q)$.
- (2) The set of the maps in general position the degree d forms open and everywhere dense subset in Rat_d .

Proof. See [B]

Proposition 29. The following cases are equivalent

- (1) The rational map R is hyperbolic.
- (2) There exists $N > 1$ such that R^N is J -stable in the space Rat_{d^N} .
- (3) For any $N > 1$ the map R^N is J -stable in the space Rat_{d^N} .

Proof. The proof of this proposition is extremely trivial. The (1) implies (2) and (3).

Now let $N > 1$ be the number from the conditions (1) or (2) of the proposition. By the lemma above there exists a map $Q \in \text{Rat}_{d^N}$ which is in the general position and a homeomorphism $\phi : J(R) \rightarrow \mathbb{C}$, so that $Q = \phi \circ R \circ \phi^{-1}$. Assume that there exists a critical point $c \in J(R)$, then only one possibility can occur $R^{-N}(c) = c$ and $\deg(R) = 2$. That means c is periodic superattractive point and cannot belong to the Julia set. Contradiction with assumption.

Note that in non-hyperbolic case J -stability in Rat_d means that the dimension of the space of fixed points for Beltrami operator should be equal to the number of critical points on the Julia set and this last number grows exponentially with respect to iterations. Indeed the dimension of the space of fixed points for Beltrami operator is equal to the number of *critical values* on and this number grows linearly with respect to iterations.

Probably the following assumption: R^n is J -stable in $H(R^n)$ for any $n > 1$ implies the hyperbolicity of R .

Now let R be a rational map, assume that R^k is J -stable in $H(R^k)$, is it true that R^n is J -stable in $H(R^n)$, for $n > k$? What is possible to say in the case $n < k$?

Remark 30. *Note that the question above are also the questions about point spectrum of Beltrami operator, that is if R^k is J -stable in $H(R^k)$, then the Beltrami operator B_R has an eigenvalue which is k -root of unity.*

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